Controllability
Monday, June 22nd 2015
Solutions

Solution Problem 1 This is a linear control system and there is no constraint on the control bound. Hence, we just need to verify that the Kalman condition holds. In this problem, $n = 6$ so we need to verify that $\text{rank}(B, AB, \ldots, A^5B) = 6$. Notice that the matrix B has the form

$$B = \begin{pmatrix} c_1 & -c_3 & 0 \\ c_2 & 0 & 0 \\ 0 & 0 & 0 \\ c_1 & c_3 & 0 \\ 0 & c_4 & 0 \\ 0 & 0 & c_5 \end{pmatrix}$$

where $c_1,\ c_2,\ c_5 \neq 0$. Computing, we get

$$AB = \begin{pmatrix} \lambda_1 c_1 & -\lambda_1 c_3 & 0 \\ 0 & c_4\omega_1 & 0 \\ 0 & 0 & c_5\omega_2 \\ -\lambda_1 c_1 & -\lambda_1 c_3 & 0 \\ -c_2\omega_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A^2B = \begin{pmatrix} \lambda_1^2 c_1 & -\lambda_1^2 c_3 & 0 \\ -c_2\omega_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_1^2 c_1 & \lambda_1^2 c_3 & 0 \\ 0 & -c_4\omega_1^2 & 0 \\ 0 & 0 & c_5\omega_2^2 \end{pmatrix},$$

and

$$A^3B = \begin{pmatrix} \lambda_1^3 c_1 & -\lambda_1^3 c_3 & 0 \\ 0 & -c_4\omega_1^3 & 0 \\ 0 & 0 & c_5\omega_2^3 \\ -\lambda_1 c_1^3 & -\lambda_1^3 c_3 & 0 \\ c_2\omega_1^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For instance, consider the first and third columns of B, the first and third columns of AB, the first columns of A^2B and the first column of A^3B. They form the set of vectors

$$\begin{pmatrix} c_1 \\ c_2 \\ 0 \\ c_1 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_5 \end{pmatrix},\ \begin{pmatrix} \lambda_1 c_1 \\ 0 \\ 0 \\ -\lambda_1 c_1 \\ -c_2\omega_1 \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ 0 \\ c_5\omega_2 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} \lambda_1^2 c_1 \\ -c_2\omega_1^2 \\ 0 \\ \lambda_1^2 c_1 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} \lambda_1^3 c_1 \\ 0 \\ 0 \\ -\lambda_1^3 c_1 \\ c_2\omega_1^3 \\ 0 \end{pmatrix}.$$

and are linearly independent if the vectors

$$u_1 = \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ c_1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} \lambda_1 c_1 \\ 0 \\ 0 \\ -\lambda_1 c_1 \\ -c_2\omega_1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} \lambda_1^2 c_1 \\ -c_2\omega_1^2 \\ 0 \\ \lambda_1^2 c_1 \\ 0 \\ 0 \end{pmatrix}, u_4 = \begin{pmatrix} \lambda_1^3 c_1 \\ 0 \\ 0 \\ -\lambda_1^3 c_1 \\ c_2\omega_1^3 \\ 0 \end{pmatrix}.$$

are linearly independent. Let $\alpha_i,\ i = 1, \ldots, 4$ be for real numbers such that $\sum_{i=1}^4 \alpha_i u_i = 0$. From row 2, we get $\alpha_1 = \alpha_3\omega_1^2$. Summing rows 1 and 4, we get $\alpha_1 = -\alpha_3\lambda_1^2$. Hence $\alpha_1 = \alpha_3 = 0$. Therefore, row 1 simplifies to $\alpha_2 = -\alpha_4\lambda_1^2$. From row 5, we get $\alpha_2 = \alpha_4\omega_1^2$. Hence $\alpha_2 = \alpha_4 = 0$. As the result, we have found 6 column vectors of the matrix $\text{rank}(B, AB, \ldots, A^5B)$ which are linearly independent so $\text{rank}(B, AB, \ldots, A^{n-1}B) = 6$.

Solution Problem 2 Prove (2) $\Leftrightarrow$ (3). (2) $\Rightarrow$ (3) is obvious so we just need to prove (3) $\Rightarrow$ (2). Let $\lambda \in \mathbb{C}$. If $\lambda \in \text{Spec}(A)$, then $\text{rank}(\lambda I - A, B) = n$ according to (3). If $\lambda \notin \text{Spec}(A)$ then $\det(\lambda I - A) \neq 0$ which is equivalent to $\text{rank}(\lambda I - A) = n$ and therefore $\text{rank}(\lambda I - A, B) = n$ as well.

Prove (3) $\Leftrightarrow$ (4). Assume (3). Let z be an eigenvector of A^T. There exists $\lambda \in \text{Spec}(A)$ such that $(\lambda I - A^T)z = (\lambda I - A)^T z = 0$. If $B^T z = 0$, then $(\lambda I - A, B)^T z = 0$ so $\text{Ker}(\lambda I - A, B)$ is not $\{0\}$ and $\text{rank}(\lambda I - A, B) < n$ which is contradictory with (3). Now, assume (4). Let $\lambda \in \text{Spec}(A)$ and $z \neq 0$. If $(\lambda I - A)^T z \neq 0$ then $(\lambda I - A, B)^T z \neq 0$. If $(\lambda I - A)^T z = 0$ then, according to (4), $(\lambda I - A, B)^T z \neq 0$. So $\text{Ker}(\lambda I - A, B)$ is $\{0\}$ and $\text{rank}(\lambda I - A, B) = n$

Prove (2) $\Leftrightarrow$ (5). Assume (5). Let $z \neq 0 \in \mathbb{R}^n$. Then $||(\lambda I - A, B)^T z||^2 > 0$ so $(\lambda I - A, B)^T z \neq 0$. Hence $\text{Ker}(\lambda I - A, B)$ is $\{0\}$ and $\text{rank}(\lambda I - A, B)^T = \text{rank}(\lambda I - A, B) = n$. Now, assume (2). If (5) does not hold then, for all $c > 0$, there exist $\lambda \in \mathbb{C}$ and $z \in \mathbb{R}^n$ such that $||(\lambda I - A^T)z||^2 + ||(B^T z||^2 < c||z||^2$. Therefore, there exists $\lambda \in \mathbb{C}$ such that $||(\lambda I - A, B)^T|| = 0$ which means that $(\lambda I - A, B)^T$ is the null operator so $\text{rank}(\lambda I - A, B)^T = \text{rank}(\lambda I - A, B) \neq n$ which is contradictory with (2).

Prove (1) $\Leftrightarrow$ (4). Assume (4) does not hold. Then there exist an eigenvector z of A^T, associated with an eigenvalue λ, such that $B^T z = 0$. Hence, for all $1 \leq k \leq n-1$, we have $(A^k B)^T z = B^T (A^T)^k z = \lambda^k B^T z = 0$. Therefore $(B, AB, \ldots, A^{n-1}B)^T z = 0$ so $\text{Ker}(B, AB, \ldots, A^{n-1}B)^T$ is not $\{0\}$ and $\text{rank}(B, AB, \ldots, A^{n-1}B)^T = \text{rank}(B, AB, \ldots, A^{n-1}B) < n$ so (1) does not hold. Now, assume that (1) does not hold. Set $N = \{z \in \mathbb{R}^n \mid z^T A^k B = 0\ \forall k \in \mathbb{N}\}$. If $z \in N$ then, for all $k \in \mathbb{N}$, $(A^T z)^T A^k B = z^T (A^T)^{k+1} B = 0$. Therefore $A^T N \subset N$. In other words, N is stable under A^T. Since (1) does not hold, there exists $z \in \mathbb{R}^n \neq 0$ such that $(B, AB, \ldots, A^{n-1}B)^T (z) \neq 0$ which means that $(A^T)^k z \in \text{Ker}(B^T)$ for all $1 \leq k \leq k-1$. Using, for instance, an induction, we can show that, necessarily, $(A^T)^k z \in \text{Ker}(B^T)$ for all $k \in \mathbb{N}$. Hence $N \neq \{0\}$. Since A^T is diagonalizable over $\mathbb{C}$, the invariant subspaces of A^T are spanned by a subset of the eigenvectors of A^T. Therefore, there is an eigenvector of A^T contained in N and (4) does not hold.

Solution Problem 3

1. When $u = 0$, the system writes

$$\begin{pmatrix} \dot{\omega_1}(t) \\ \dot{\omega_2}(t) \\ \dot{\omega_3}(t) \end{pmatrix} = \begin{pmatrix} \omega_2(t)\omega_3(t) \\ -\omega_1(t)\omega_3(t) \\ \omega_1(t)\omega_2(t) \end{pmatrix} = F_0 \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

The vector field F_0 is volume-preserving since $\text{div}F_0 = \sum_{i=1}^{3} \frac{\partial F_0}{\partial \omega_i} = 0$. In addition, notice that the mapping $f(\omega_1, \omega_2, \omega_3) = \omega_1^2 + 2\omega_2^2 + \omega_3^2$ is constant along every solution of the system since we have $\frac{d}{dt} f(\omega_1(t), \omega_2(t), \omega_3(t)) = 0$. Therefore, every ellipsoid $\omega_1^2 + 2\omega_2^2 + \omega_3^2 = c,\ c > 0$ is invariant under the flow of F_0. Since such ellipsoid are all bounded, the Poincaré recurrence theorem asserts that F_0 is positively recurrent/Poisson-stable on every ellipsoid. Hence, F_0 is Poisson-stable.

2. We denote

$$F_1 \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The system is controllable if and only if $\text{Lie}_\omega\{F_0, F_1\}$ has rank 3 for all $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$. In other words, we want to show that the rank condition is satisfied. Notice that F_0 is a degre 2 homogeneous polynomial vector field. Indeed, for all $\alpha \in \mathbb{R}$, we have Lie_ω

$$F_0(\alpha\omega) = \alpha^2 F_0(\omega).$$

Therefore $\text{Lie}_\omega\{F_0, F_1\} = \mathbb{R}^3$ for all $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ if and only if the subalgebra of constant vector fields in $\text{Lie}\{F_0, F_1\}$ is of dimension 3 (classic theorem, see for instance Geometric control theory, V. Jurdjevic, Cambridge Studies in Advanced Mathematics, p143). A basis for this subalgebra can be found among the Lie brackets of $\{F_0, F_1\}$ of order up to 4 (another classic theorem, see again Geometric control theory, V. Jurdjevic, Cambridge Studies in Advanced Mathematics, p144). Computing, we find that this subalgebra is spanned by the constant vector fields

$$\begin{aligned} f_1 &= b \\ f_2 &= [[F_0, f_1], f_1], \\ f_3 &= [[F_0, f_1], f_2], \\ f_4 &= [[F_0, f_1], f_3], \\ f_5 &= [[F_0, f_2], f_2], \end{aligned}$$

and the rank is 3 if and only if the 3 vectors

$$\begin{aligned} f_1 &= (b_1, b_2, b_3)^T, \\ f_2 &= 2(b_2b_3, -b_1b_3, b_1b_2)^T, \\ f_3 &= 2(b_1(-b_3^2 + b_2^2), -b_2(b_3^2 + b_1^2), b_3(b_2^2 - b_1^2))^T \end{aligned}$$

are linearly independant. Therefore, the rank condition is satisfied everywhere on $\mathbb{R}^3$, unless two of the entries $b_i,\ i = 1, 2, 3$ are zeros or $b_3 = \pm b_1$.